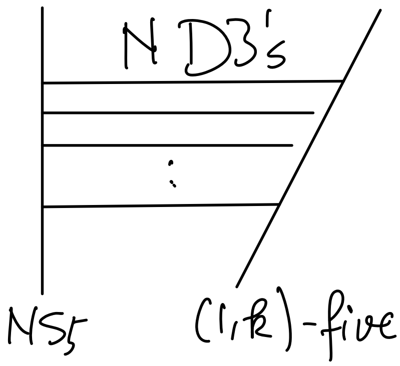
Last time we saw that compactification of 6d (2,0) - theory of N M5 branes on the lense space $L(k,1)$ is equiv. to the following brane setup:



spacetime	0	1	2	3	4	5	6	7	8	9
N D3's	-	-	-	.	.	.	H	.	.	.
NS5	-	-	-	-	-	-	.	.	.	.
(1,k)-brane	-	-	-	-	-	-	\	.	.	\

One can show that this leads to 3d $\mathcal{N}=2$ $U(N)$ super-CS-th at level k + 1 massive chiral multiplet Φ in adjoint representation

$\longrightarrow$ $U(1)_\beta$ rotates x^3 and x^4

$\longrightarrow$ background exp. value gives real mass β to Φ:

$$\delta S_{mass} = \int d^3x\, d^4\theta\, \Phi e^{\beta\theta^2} \Phi^\dagger$$

To summarize, we have arrived at the following correspondence:

3d $\mathcal{N}=2$-th $T[L(k,1);\beta]$	$\longleftrightarrow$	3d $\mathcal{N}=2$-th $T[\Sigma\times S^1;\beta]$
SCS-th at level k with adjoint Φ of mass β		sigma-model with target $\mathcal{M}_H$ and a real mass for $U(1)_\beta$ flavor sym.
$\downarrow$ evaluate partition function on $\Sigma\times S^1$		$\downarrow$ evaluate p.f. on $L(k,1)$

$$Z_{T[L(k,1);\beta]}(\Sigma\times S^1) = Z_{T[\Sigma\times S^1;\beta]}[L(k,1);SL(N,\mathbb{C})]$$

$$= \dim_\beta \mathcal{H}(\Sigma; G_{\mathbb{C}}, k)$$

"Equivariant Verlinde Formula"

$$= \sum_n t^n \dim \mathcal{H}_n$$

where $t=e^{-\beta}$

The degree-0 piece is given by

$$\mathcal{H}_0 = \mathcal{H}(\Sigma; G, k) = \#(\text{conformal blocks on } \Sigma)$$

$$= \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin\frac{\pi j}{k+2}\right)^{2-2g}$$

"Verlinde Formula"

Equivariant Integration over Hitchin moduli space

1) Quantization of Hitchin moduli space:

$$\mathcal{M}_{flat}(\Sigma; G) = \{A \mid F_A = 0\}/\mathcal{G}$$

→ equipped with symplectic form

$$\omega = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\, \delta A \wedge \delta A$$

where δ is de Rham diff. on $\mathcal{M}_{flat}$

⟶ ω generator of $H^2(\mathcal{M}_{flat}; \mathbb{Z})$

→ classical phase space of CS-th: $(\mathcal{M}_{flat}(\Sigma; G), k\omega)$

→ geometric quantization identifies $\mathcal{H}_{CS}(\Sigma; G)$ with hol. sections of line bundle $\mathcal{L}^{\otimes k}$:

$$\mathcal{H}_{CS}(\Sigma; G, k) = H^0(\mathcal{M}_{flat}(\Sigma; G), \mathcal{L}^{\otimes k})$$

Kodaira vanishing

$$= \chi(\mathcal{M}_{flat}, \mathcal{L}^{\otimes k}) = \mathrm{Index}(\not{D}_{\mathcal{L}^{\otimes k}})$$

$$= \int_{\mathcal{M}_{flat}} \underset{\text{"Todd class" of } \mathcal{M}_{flat}(\Sigma;G)}{\underbrace{Td(\mathcal{M}_{flat})}} \wedge e^{k\omega}$$

Now let us consider Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$

$\longrightarrow$ classical phase space becomes:

$$\left(\mathcal{M}_{flat}(\Sigma; G_{\mathbb{C}}) \simeq \mathcal{M}_H(\Sigma; G), k\omega_I + \sigma\omega_K\right)$$

where

$$\mathcal{M}_H(\Sigma; G_{\mathbb{C}}) = \left\{ (A, \phi) \,\middle|\, \begin{array}{l} F_A - \phi\wedge\phi = 0 \\ d_A\phi = d_A^{\dagger}\phi = 0 \end{array} \right\} \Big/ \mathcal{G}$$

with $\phi \in \Omega^1(\Sigma, \mathfrak{g})$ (imaginary part of complex gauge field)

$\mathcal{M}_H$ is hyper-kähler

$\rightarrow$ equipped with 3 complex structures

$$\omega_I = \frac{1}{4\pi^2} \int_\Sigma Tr(\delta A \wedge \delta A - \delta\phi\wedge\delta\phi)$$

$$\omega_J = \frac{1}{2\pi^2} \int_\Sigma Tr(\delta A \wedge * \delta\phi)$$

$$\omega_K = \frac{1}{2\pi^2} \int_\Sigma Tr(\delta A \wedge \delta\phi)$$

→ is complexification of $\mathcal{M}_{flat}(\Sigma; G)$ birationally equivalent to $T^*\mathcal{M}_{flat}$

→ quantization gives:

$$\dim \mathcal{H}_{CS}(\Sigma; G_{\mathbb{C}}, k) = \int_{\mathcal{M}_H} \mathrm{Td}(\mathcal{M}_H) \wedge e^{k\omega_I + \sigma\omega_K}$$

non-compact

→ integral divergent

<u>But</u>: $\mathcal{M}_H$ admits $U(1)$ action with compact fixed point loci

→ denote by $U(1)_\beta$

The corresponding vector field on $\mathcal{M}_H$, denoted by V, is generated by the Hamiltonian:

$$\mu = \frac{1}{2\pi} \int_\Sigma \mathrm{Tr}(\phi \wedge * \phi)$$

$$\delta\mu = 2\pi \, \iota_V \omega_I$$

→ define equivariant integral:

$$\int_{\mathcal{M}_H} \mathrm{ch}(\mathcal{L}^{\otimes}) \wedge \mathrm{Td}(\mathcal{M}_H) \rightsquigarrow \int_{\mathcal{M}_H} \mathrm{ch}(\mathcal{L}^{\otimes}, \beta) \wedge \mathrm{Td}(\mathcal{M}_H, \beta)$$

where the equivariant Chern character is given by

$$\mathrm{ch}(\mathcal{L}^{\otimes k}, \beta) = \exp(k\,\tilde{\omega}_I) = \exp(k\omega_I - k\beta\mu)$$

$\rightarrow$ exponentially suppresses the contributions away from $\mathcal{M}_{flat}(\Sigma; G) \subset \mathcal{M}_H$ $(\mu \gg 0)$

$\rightarrow$ Atiyah-Bott localization gives

$$\mathrm{Index}_{S^1}(\partial\!\!\!/_{\mathcal{L}^{\otimes k}}, \beta) = \sum_{F_d} e^{-\beta k \cdot \mu(F_d)} \int_{F_d} \frac{Td(F_d) \wedge e^{k\omega_I}}{\prod_i (1 - e^{-x_i - \beta n_i})} \quad (*)$$

F_d: critical loci of μ

Can be either computed directly or by using the duality above!

We will proceed along the second path

β-deformed complex CS-th

The 3d $\mathcal{N}=2$ theory $T[L(k,1);\beta]$ can be twisted on $\Sigma \times S^1$

→ resulting theory is "β-deformed $G_{\mathbb{C}}$ complex Chern-Simons theory" at level k

Has the following properties:

1) For $\beta \to +\infty$ it reduces to CS-th. with compact gaug group G at level k

2) For $\beta \to 0$ it becomes Chern-Simons theory with non-compact gauge group $G_{\mathbb{C}}$

3) For general β, it reproduces the equivariant integral $(*)$ over the Hitchin moduli space $\mathcal{M}_H$ (if put on $\Sigma \times S^1$)

→ We are interested in sector 3)

Equivariant G/G gauged WZW model

The partition function of $T[L(k,1);\beta]$ on $S^1 \times \Sigma$ is equivalent to the one of equivariant gauged WZW model on Σ.

$\rightarrow$ in the limit $\beta \to 0$ obtain ordinary gauged WZW model on Σ

Fields are:

- (A, λ, g) where A is gauge field, $g \in \mathcal{G} \cong \mathrm{Map}(\Sigma, G)$, and λ is auxiliary Grassmann 1-form in adjoint rep.
- at level k, the action of the G/G model is

$$k\, S_{G/G}(A,\lambda,g) = k\, S_G(A,g) - ik\,\Gamma(A,g) + \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}(\lambda \wedge \lambda)$$

where $$S_G(g,A) = -\frac{1}{8\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1} d_A g \wedge * g^{-1} d_A g\right)$$

and

$$\Gamma(g,A) = \frac{1}{12\pi}\int_B \mathrm{Tr}\left[(g^{-1}dg)^3\right]$$

$$-\frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left(A\,dg\,g^{-1} + AA^g\right)$$

where B is handlebody with $\partial B = \Sigma$